Transformation from the general form of state-space to another form "Review"

general form:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

old form → o/p

$x = Tz$ (linear transformation)

$$\dot{z} = \bar{A}z + \bar{B}u$$
$$y = \bar{C}z$$

→ New form

the two forms ~~are~~ have both the same ch/equation

given : A, B, C

find T ⇒ transformation matrix

and $\bar{A}$, $\bar{B}$, $\bar{C}$

$\therefore x = Tz$ (linear transformation)

$x(t) = T z(t)$

$\dot{x}(t) = T \dot{z}(t)$

$\dot{x} = Ax + Bu$

$T\dot{z}(t) = ATz + Bu \Rightarrow \dot{z}(t) = T^{-1}ATz + T^{-1}Bu$

$\therefore \bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$

$y = Cx = CTz$, $y = CTz \Rightarrow \bar{C} = CT$

note: T "transformation matrix" depend on the form we need to it

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"Review"

[1] Transformation into diagonal form :

this mean that the original form isn't the diagonal form

assume second order system :

$$\bar{A} = T^{-1}AT = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

, $s_1, s_2 \Rightarrow$ poles

" ~~controllable~~ diagonal form "

let $T = (V_1 \quad V_2)$

, $V_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, $V_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

$$(s_i I - A) V_i = 0 \qquad , i = 1, 2.$$

$(s_1 I - A) V_1 = 0$, $(s_2 I - A) V_2 = 0$

poles poles

$$\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} = T^{-1} A T \Rightarrow T \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} = AT$$

$$(V_1 \quad V_2) \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} = AT \Rightarrow (V_1 \; V_2) \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} - AT = 0$$

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Ex.

$$\dot{x} = \begin{pmatrix} -5 & -1 \\ 3 & -1 \end{pmatrix} x + \begin{pmatrix} 2 \\ 5 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 2 \end{pmatrix} x$$

transfer to diagonal form.

Sol.

poles: $|SI - A| = 0 \quad \left| \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} - \begin{pmatrix} -5 & -1 \\ 3 & -1 \end{pmatrix} \right| = 0$

$$\begin{vmatrix} S+5 & 1 \\ -3 & S+1 \end{vmatrix} = 0 \longrightarrow (S+5)(S+1) + 3 = 0$$

$$S^2 + 6S + 8 = 0$$

∴ the system poles $S_{1,2} = -2, -4$

① $i=1 \Rightarrow S_1 = -2$ (first pole)

$$\begin{pmatrix} S_i + S & 1 \\ -3 & S_i + 1 \end{pmatrix} V_i = 0 \qquad \text{at } i=1 \quad \begin{pmatrix} S_1 + S & 1 \\ -3 & S_1 + 1 \end{pmatrix} V_1 = 0$$

$$\therefore \begin{pmatrix} -2+S & 1 \\ -3 & -2+1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0 \Rightarrow \boxed{\begin{matrix} 3a_1 + b_1 = 0 \\ -3a_1 - b_1 = 0 \end{matrix}} \Longrightarrow \text{the same equation}$$

assume: $\boxed{a_1 = 1} \Rightarrow \boxed{b_1 = -3}$

② $i=2 \Rightarrow S_2 = -4$ (second pole)

$$\begin{pmatrix} S_2 + S & 1 \\ -3 & S_2 + 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -4+S & 1 \\ -3 & -4+1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0$$

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$$\begin{pmatrix} 1 & 1 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} a_2 \\ b_1 \end{pmatrix} = 0 \Rightarrow \begin{matrix} a_2 + b_2 = 0 \\ -3a_2 - 3b_2 = 0 \end{matrix}$$ → the same equations

assume $\boxed{a_2 = 1}$, $\boxed{b_1 = -1}$

$$\therefore T = \begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}$$

$$\bar{A} = T^{-1} A T = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}$$

$$\bar{A} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -2 & -4 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 3 & 2 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\bar{B} = T^{-1} B = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3.5 \\ 5.5 \end{pmatrix}$$

$$\bar{C} = TC = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -1 \end{pmatrix}$$

$$\therefore \dot{z} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} z + \begin{pmatrix} -3.5 \\ 5.5 \end{pmatrix}$$

~~ex~~ $y = \begin{pmatrix} -5 & -1 \end{pmatrix} z$

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ex:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$y = (1 \quad 2)\, x$ transform into diagonal form

(Sol.)

Poles $\Rightarrow$ ch. eq $\Rightarrow |ST - A| = 0 \Rightarrow \begin{vmatrix} S & -1 \\ 2 & S+3 \end{vmatrix} = 0$

Ch. eq : $S^2 + 3S + 2 = 0 \Longrightarrow$ poles $\Rightarrow S_1 = -1, \quad S_2 = -2$

$(S_i I - A) V_i = 0$

for $i = 1 \Rightarrow S_1 = -1 \quad \& V_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

$$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = 0$$

$-a_1 - b_1 = 0 \quad , \quad 2a_1 + 2b_1 = 0 \Longrightarrow$ the same equation

assume $a_1 = 1 \Rightarrow b_1 = -1 \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

for $i = 2 \Rightarrow S_2 = -2 \Rightarrow V_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

$$\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} -2a_2 - b_2 = 0 \\ 2a_2 + b_2 = 0 \end{matrix}$$

assume $a_2 = 1 \longrightarrow b_2 = -2 \Rightarrow V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\bar{A} = T^{-1} A T = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \bar{B} = T^{-1} B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} -1 & -3 \end{pmatrix} = CT$$

Special case: A in controllable: #

$$T_{2\times 2} = \begin{pmatrix} 1 & 1 \\ S_1 & S_2 \end{pmatrix}, \quad T_{3\times 3} = \begin{pmatrix} 1 & 1 & 1 \\ S_1 & S_2 & S_3 \\ S_1^2 & S_2^2 & S_3^2 \end{pmatrix}$$

$S_1, S_2 \Rightarrow$ poles

$S_1, S_2, S_3 \Rightarrow$ poles

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If the poles is repeated

$$T_{3\times3} = \begin{pmatrix} 1 & 0 & 1 \\ s_1 & 1 & s_2 \\ s_1^2 & 2s_1 & s_2^2 \end{pmatrix}$$

$s_1, s_1, s_2 \Rightarrow$ poles

State feedback control / pole placement design

one of the modern control design approach

* Model control design

⇒ The system model based on state space model.

Control → classification depend on i/p

- servo problem design / control ($r \neq 0$) → input
- regulation problem design / control ($r = 0$) → state-feedback control

Regulation problem design ($r = 0$)
when the control is done to reject the disturbance of the system, this process is called regulation control.

Servo problem design ($r \neq 0$)
when changing the setting point (~~desig~~ desired o/p).

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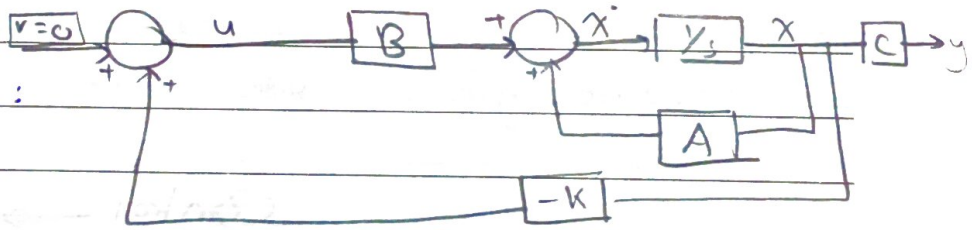
The o/p match this setting point, this process is called servo control.

SO :-

State feedback control / pole placement design :-
is a - regulation problem
- Modern control design

given :- $\dot{x} = Ax + Bu$, control law :- $u = -\underbrace{K}_{\text{gain matrix}} x$
$y = Cx$

the design is to find the gain K



To find the gain K :

$\dot{x} = Ax + Bu$
$y = Cx$
$u = -Kx$

$\therefore \dot{x} = Ax - BKx$
$\therefore \dot{x} = (A - BK)x$

the desired ch. equation

$$|SI - (A - BK)| = 0 \longrightarrow (1)$$

given design specs.
(the location of the desired poles)

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Second degree

given : P_1 , P_2 (Poles)

The ~~desig~~ desired ch. equation $\alpha_c(s)$

$$\alpha_c(s) = (s-P_1)(s-P_2) = 0 \quad (2)$$

from (1) and (2) we can obtain the gain K

$$K = [K_1 \quad K_2 \quad K_3 \quad \dots \quad K_n]$$

↑ system order

example:

$$\dot{x} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

$$y = (1 \quad -1)\, x$$

design a state feedback controller to locate place the closed loop poles at $S = -1 \pm j2$

Sol:

(1) desired ch/eq $\alpha_c(s)$

$$\alpha_c(s) = (S+1-j2)(s+1+j2) = 0$$

$$(S+1)^2 + 4 = 0 \Rightarrow \boxed{S^2 + 2S + 5 = 0} \quad (1)$$

(2) the desired ch/eq after using contol law $u = -Kx$

$$|SI - (A - BK)| = 0$$

$$A - BK = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}(K_1 \quad K_2) = \begin{pmatrix} 2-K_1 & -K_2 \\ 1 & 0 \end{pmatrix}$$

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$$\left| \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 2-K_1 & -K_2 \\ 1 & 0 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} s-2+K_1 & +K_2 \\ -1 & s \end{vmatrix} = 0 \Rightarrow s^2 - (2-K_1)s + K_2 = 0$$

$$\boxed{s^2 + (K_1 - 2)s + K_2 = 0} \rightarrow (2)$$

(1) = (2) $\longrightarrow K_1 - 2 = 2 \longrightarrow \boxed{K_1 = 4}$ #

$\boxed{K_2 = 5}$ #

$$\therefore K = [\,4 \quad 5\,] \;\#$$

Note: check Controllability befour the solution
if the system is not controllable then
~~we can~~ no solution for this problem

$M_c = (B \quad AB)$

, $|M_c| \neq 0$ (controllable)

another solution:

Ackermann Method: → 2nd order

$K = [K_1 \quad K_2] = (0 \quad 1)\, M_c^{-1}\, \alpha_c(A)$

, $\alpha_c(A) = \alpha_c(s)\big|_{s \to A}$

→ desired ch. equation

If the system is third order

$K = [K_1 \quad K_2 \quad K_3] = [0 \quad 0 \quad 1]\, M_c^{-1}\, \alpha_c(A)$

the solution :-

$M_c = (B \quad AB), \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$|M_c| = (1 - 0) = 1 \neq 0$

$K = (0 \quad 1)\, M_c^{-1}\, \alpha_c(A)$

$M_c^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$\alpha_c(s) = s^2 + 2s + 5 = 0$

$\alpha_c(A) = A^2 + 2A + 5I = 0$

$= \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}^2 + 2 \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$\alpha_c(A) = \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 4 & 5 \end{pmatrix}$

$\therefore K = (0 \quad 1) \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 13 & 0 \\ 4 & 5 \end{pmatrix} = (0 \quad 1) \begin{pmatrix} 13 & 0 \\ 4 & 5 \end{pmatrix}$

$K = (\underset{K_1}{4} \quad \underset{K_2}{5})$ #

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Example

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

determine the feed back gain K to make

$\zeta_o = 0.5$ & $t_s = 2$ sec

Solution

$$t_{s=2} = \frac{4}{\omega_n \zeta} \longrightarrow \omega_n \zeta = 2$$

$\therefore \omega_n = 4$

$$s_{1,2} = -2 \pm j4\sqrt{1-0.25}$$

$$= -2 \pm j3.46$$

Report